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THE CLEBSCH-GORDAN PROBLEM FOR THE POLYNOMIAL ALGEBRA

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1. Introduction

This report is a survey of the results that I presented at RIMS, Kyoto during the conference *Expansion of Combinatorial Representation Theory.* The proofs of these results can be found in a joint paper with Erik Darpö [3].

The Clebsch-Gordan problem has its origin in the invariant theory of Clebsch and Gordan. In [2] Clebsch investigates algebraic forms. One of his results was later understood as the solution to the following problem [8]: given two indecomposable representations V and W of SL_2, find the decomposition of their tensor product into a direct sum of indecomposables,

$$V \otimes W \xrightarrow{\sim} \bigoplus_{i \in I} U_i.$$

This problem, called the Clebsch-Gordan problem can be posed for any group, or more generally, for any Krull-Schmidt category equipped with a tensor product. In this article we are interested in the category of finite dimensional modules over the polynomial algebra. Object in this category correspond to linear operators and the tensor product to the tensor product of linear operators. One motivation for this problem is that it appears in the construction of graded Frobenius algebras by Wakamatsu in [9], as pointed out in [6]. We start our investigation with a very classical case.

Let $\Bbbk$ be a field. For $\lambda \in \Bbbk$ and l a positive integer, let $J_\lambda(l)$ be the Jordan block of size l and eigenvalue λ:

$$J_\lambda(l) = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$$

For any two matrices A and B we write $A \sim B$ if A is conjugate to B i.e., if $A = TBT^{-1}$ for some invertible matrix T.

If $\Bbbk$ is algebraically closed, then every square matrix is conjugate to its Jordan normal form. Hence the Clebsch-Gordan problem amounts to the following: find the Jordan normal form of the Kronecker product of two Jordan blocks $J_\lambda(l) \otimes J_\mu(m)$ for all $\lambda, \mu \in \Bbbk$ and positive integers l, m. In characteristic zero this problem was solved quite early by Aitken [1]. There are also independent solutions by Huppert [5], and by Martsinkovsky and Vlassov [7]. The solution is given by the following Theorem.

Theorem 1. *For all $\lambda, \mu \in \Bbbk \setminus \{0\}$ and positive integers l, m the following fomulae hold:*

(1) $J_\lambda(l) \otimes J_\mu(m) \sim \bigoplus_{i=0}^{l-1} J_{\lambda\mu}(l+m-2i-1)$ *if* $l \leqslant m$ *and* $\operatorname{char} \Bbbk = 0$,
(2) $J_\lambda(l) \otimes J_0(m) \sim lJ_0(m)$,
(3) $J_0(l) \otimes J_0(m) \sim (m-l+1)J_0(l) \oplus \bigoplus_{i=1}^{l-1} 2J_0(i)$ *if* $l \leqslant m$.

In positive characteristic the first formula of Theorem 1 does not hold. However, one may observe that for $\lambda \neq 0$,

$$J_\lambda(l) \sim J_\lambda(1) \otimes J_1(l)$$

and thus for $\mu \neq 0$,

$$J_\lambda(l) \otimes J_\mu(m) \sim J_\lambda(1) \otimes J_\mu(1) \otimes J_1(l) \otimes J_1(m) \sim J_{\lambda\mu}(1) \otimes J_1(l) \otimes J_1(m).$$

Hence it suffices to find the Jordan normal form of $J_1(l) \otimes J_1(m)$, which will only contain Jordan blocks of eigenvalue 1 since $J_1(l) \otimes J_1(m)$ is upper triangular with all diagonal elements equal to 1. For $\Bbbk$ of prime characteristic there is an algorithm for finding this Jordan normal form, discovered by Iima and Iwamatsu [6]. However, no explicit formula is known.

Throughout the rest of the paper, assume that $\Bbbk$ is perfect i.e., all irreducible polynomials over $\Bbbk$ have distinct zeros. In this case not every linear operator can be put on Jordan normal form and we have to adjust the set-up of the problem accordingly.

2. Main problem

Let V and W be modules over the polynomial algebra $\Bbbk[x]$. Then their tensor product over $\Bbbk$ has the structure of a $\Bbbk[x]$-module with $x(v \otimes w) = xv \otimes xw$ for all $v \in V$ and $w \in W$. Together with this tensor product comes the Clebsch-Gordan problem, which if $\Bbbk$ is algebraically closed, is to the problem described above. To proceed we recall the classification of indecomposable $\Bbbk[x]$-modules.

Theorem 2. *The modules $\Bbbk[x]/f(x)^s$, where s is a positive integer and $f(x) \in \Bbbk[x]$ is irreducible and monic, classify all indecomposable finite-dimensional $\Bbbk[x]$-modules up to isomorphism.*

To solve the Clebsch-Gordan problem for $\Bbbk[x]$ we need to decompose $\Bbbk[x]/f(x)^s \otimes \Bbbk[x]/g(x)^t$ for all positive integers s, t and irreducible monic polynomials $f(x), g(x)$.

The two second formulae of Theorem 1 generalise in the following way.

Proposition 1. *Let s and t be positive integers and $f(x) \in \Bbbk[x]$ irreducible with $f(0) \neq 0$. Then the following formulae hold.*

(1) $\Bbbk[x]/x^s \otimes \Bbbk[x]/f(x)^t \xrightarrow{\sim} t(\deg f)\Bbbk[x]/x^s$.
(2) $\Bbbk[x]/x^s \otimes \Bbbk[x]/x^t \xrightarrow{\sim} (t-s+1)\Bbbk[x]/x^s \oplus \bigoplus_{i=1}^{s-1} 2\Bbbk[x]/x^i$ *if* $s \leqslant t$.

It remains to decompose $\Bbbk[x]/f(x)^s \otimes \Bbbk[x]/g(x)^t$ for all positive integers s, t and irreducible polynomials $f(x), g(x) \in \Bbbk[x]$ such that $f(0) \neq 0 \neq g(0)$. For $\Bbbk$ algebraically closed, this case concerns Jordan blocks of non-zero eigenvalues. As we saw these eigenvalues may be assumed to be equal to 1. This reduction generalises in the following way.

Lemma 1. *For any positive integer s and irreducible polynomial $f(x) \in \mathbf{k}[x]$ with $f(0) \neq 0$, the $\mathbf{k}[x]$-modules $\mathbf{k}[x]/f(x)^s$ and $\mathbf{k}[x]/(x-1)^s \otimes \mathbf{k}[x]/f(x)$ are isomorphic.*

Now the remainder of the Clebsch-Gordan problem is divided into two subproblems:

- Decompose $\mathbf{k}[x]/f(x) \otimes \mathbf{k}[x]/g(x)$ for all irreducible $f(x), g(x) \in \mathbf{k}[x]$ with $f(0) \neq 0 \neq g(0)$.
- Decompose $\mathbf{k}[x]/(x-1)^s \otimes \mathbf{k}[x]/(x-1)^t$ for all positive integers s, t.

The first subproblem has a partial solution.

Proposition 2. *Let $f(x), g(x) \in \mathbf{k}[x]$ be irreducible with $f(0) \neq 0 \neq g(0)$, and Λ and M their respective sets of zeros in $\overline{\mathbf{k}}$. If*

$$\mathbf{k}[x]/f(x) \otimes \mathbf{k}[x]/g(x) \xrightarrow{\sim} \bigoplus_{j \in J} \mathbf{k}[x]/h_j(x)$$

then the zeros in $\overline{\mathbf{k}}$ of all $h_j(x)$, counting repetitions, are precisely the numbers $\lambda\mu$, with $(\lambda, \mu) \in \Lambda \times M$.

The question remains how the zeros $\lambda\mu$ should be assembled into the polynomials $h_j(x)$. In general this seems to be a difficult problem. However, using Galois theory some things can be said. See [3] for details.

The second subproblem can be be interpreted in terms of Jordan blocks. Namely, if

$$J_1(s) \otimes J_1(t) \sim \bigoplus_{i \in I} J_1(l_i),$$

then

$$\mathbf{k}[x]/(x-1)^s \otimes \mathbf{k}[x]/(x-1)^t \xrightarrow{\sim} \bigoplus_{i \in I} \mathbf{k}[x]/(x-1)^{l_i}.$$

It turns out that this problem is related to modular group representation theory. To exploit this connection we shall change perspective to the representation ring.

3. Representation ring

The representation ring R of $\mathbf{k}[x]$, by definition has a $\mathbb{Z}$-basis consisting of the isoclasses of indecomposable $\mathbf{k}[x]$-modules. For two such isoclasses $[V]$ and $[W]$ their product is

$$[V][W] = \sum_{i \in I} [U_i],$$

where

$$V \otimes W \xrightarrow{\sim} \bigoplus_{i \in I} U_i,$$

is the decomposition of $V \otimes W$ into indecomposables. The representation ring is commutative with identity element $[\mathbf{k}[x]/(x-1)]$. Define the ring morphism

$$\dim : R \longrightarrow \mathbb{Z}$$

by $\dim[V] = \dim V$.

By Proposition 1, the $\mathbb{Z}$-span of the elements $[\Bbbk[x]/x^s]$ form an ideal I in R. Moreover, if x acts as an automorphism on a $\Bbbk[x]$-module V, then $[V]w = \dim Vw$ for all $w \in I$.

The division into subproblems described in the previous section can be reinterpreted as follows. The elements $[\Bbbk[x]/f(x)]$, where $f(x) \in \Bbbk[x]$ is irreducible and $f(0) \neq 0$ span a subring of R, which we denote by $\bar{R}$. Also, the elements $v_s := [\Bbbk[x]/(x-1)^s]$ span a subring that we denote by R'.

Now define a ring structure on $R' \otimes_{\mathbb{Z}} \bar{R} \oplus I$ by $(a \otimes b)w = \dim(a)\dim(b)w$ for all $a \in R'$, $b \in R'$ and $w \in I$. The relevance of this ring is seen in the following Theorem. Key steps of the proof are provided by Lemma 1 and Proposition 1.

Theorem 3. *The $\mathbb{Z}$-linear map*

$$\phi : R' \otimes_{\mathbb{Z}} \bar{R} \oplus I \to R,$$

defined by $\phi(a \otimes b + w) = ab + w$ is a ring isomorphism.

The multiplicative structure of the ideal I is described by Proposition 1. Moreover the elements in R' and $\bar{R}$ act on I as scalars via dim. Hence it remains to describe the rings R' and $\bar{R}$. The ring $\bar{R}$ has the following description, which follows from Proposition 2.

Proposition 3. *Let $G = \mathcal{G}(\overline{\Bbbk}/\Bbbk)$ be the absolute Galois group of $\Bbbk$ and $\overline{\Bbbk}^{\iota}$ the group of invertible elements in $\overline{\Bbbk}$. There is an isomorphism of rings:*

$$\bar{R} \xrightarrow{\sim} (\mathbb{Z}\overline{\Bbbk}^{\iota})^G$$

Where $(\mathbb{Z}\overline{\Bbbk}^{\iota})^G$ denotes the ring of invariants under G.

The ring structure of R' only depends on the characteristic of $\Bbbk$. If $\operatorname{char}\Bbbk = 0$, then we can use Theorem 1 to describe the ring R'. The result is as follows.

Theorem 4. *Assume that the characteristic of $\Bbbk$ is zero. The ring morphism*

$$\phi : \mathbb{Z}[T] \to R',$$

defined by $T \mapsto v_2$ is an isomorphism.

Now assume that $\operatorname{char}\Bbbk = p > 0$. Moreover, set $w_\alpha = v_{p^\alpha+1} - v_{p^\alpha-1}$. Then R' can be viewed as the union of the representation rings of cyclic p-groups, which have been described by Green [4]. Translating [4, Theorem 3] we obtain the following result.

Theorem 5. *Assume that* $\operatorname{char}\Bbbk = p > 0$ *and let $\alpha \in \mathbb{N}$. Set $q = p^\alpha$. Then*

$$w_\alpha v_r = \begin{cases} v_{r+q} - v_{q-r} & \text{if} \quad 1 \leqslant r \leqslant q \\ v_{r+q} + v_{r-q} & \text{if} \quad q < r \leqslant (p-1)q \\ v_{r-q} + 2v_{pq} - v_{(2p-1)q-r} & \text{if} \quad (p-1)q < r \leqslant pq \end{cases}$$

Moreover this equation defines the multiplicative structure of R'.

Theorem 5 can be used to compute the Krull-Schmidt decomposition of $\Bbbk[x]/(x-1)^s \otimes \Bbbk[x]/(x-1)^t$ or equivalently, write the product $v_s v_t$ as a linear combination of the elements v_l. First write both v_s and v_t as polynomials in the elements w_α. Then multiply the polynomials together and write the result as a linear combination of elements of the form v_l using Theorem 5.

To write v_s as a polynomial in the elements w_α first put $s = q + r$ for some $\alpha \in \mathbb{N}$ and $1 \leqslant r \leqslant (p-1)q$, where again $q = p^\alpha$. By Theorem 5,

$$v_s = \begin{cases} w_\alpha v_r + v_{s-2r} & \text{if} \quad 1 \leqslant r \leqslant q \\ w_\alpha v_r - v_{s-2q} & \text{if} \quad q < r \leqslant (p-1)q. \end{cases}$$

Applying this formula repeatedly will eventually yield v_s as a polynomial in the elements w_α.

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